# MINIMA OF FUNCTIONS OF LINES*

BY

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A function of a line or functional $F[\lambda(x)]$ is a function having as its argument an arc defined over an interval $a \leqq x \leqq b$. It may be regarded as a generalization of a function $F(\lambda_1, \cdots, \lambda_n)$ of a finite number of variables $\lambda_i$ $(i = 1, \cdots, n)$, the index $i$ with the range $1, 2, \cdots, n$ being replaced by the variable $x$ with the interval $a \leqq x \leqq b$ as its range. It is the purpose of this paper to consider some properties of a functional of this kind which has a minimum value at a particular arc $\lambda_0(x)$. As in the case of a function $F(\lambda_i)$, $(i = 1, \cdots, n)$, this involves the notion of derivatives or differentials of the first and higher orders.

Fréchet has defined first† and second‡ differentials for a function of a line $F(\lambda)$ in terms of which the difference $F(\lambda_0 + \eta) - F(\lambda_0)$ may be expressed, when $\lambda_0(x) + \eta(x)$ is a variation of $\lambda_0(x)$. The first differential is a so-called linear functional $L(\eta)$, and the second is expressible in the form $B(\eta, \eta)$, where $B(u, v)$ is a bilinear functional in the independent arguments $u(x)$, $v(x)$. Riesz has shown§ that a linear functional can always be represented as a Stieltjes integral

$$L(\eta) = \int_a^b \eta(x)\, du(x),$$

and Fréchet has deduced an analogous formula for $B(\eta, \eta)$,

$$B(\eta, \eta) = \int_a^b \int_c^d \eta(x)\eta(y)\, d_{xy}\, p(x, y),$$

in terms of a double integral which is a generalization for two dimensions of the simple Stieltjes form.

* Presented to the Society, October 30, 1920.

† M. Fréchet, *Sur la notion de différentielle d'une fonction de ligne*, these Transactions, vol. 15 (1914), p. 139. This article will be referred to as Fréchet I.

‡ M. Fréchet, *Sur les fonctions bilinéaires*, these Transactions, vol. 16 (1915), p. 232. This article will be referred to as Fréchet II.

§ F. Riesz, *Sur les opérations fonctionelles linéaires*, Comptes Rendus, 149, p. 974–977; *Démonstration nouvelle d'un théoreme concernant les opérations fonctionelles linéaires*, Annales scientifiques de l'école normale supérieure, vol. 31 (1914). The first article will be referred to as Riesz I; the second, as Riesz II.

The definitions of Fréchet apply only to functionals and differentials having continuity of order zero, which means in the case of $F(\lambda)$ that the difference $F(\lambda_0 + \eta) - F(\lambda_0)$ approaches zero with the maximum of $|\eta(x)|$ on the interval $ab$. The continuity of $F(\lambda)$ is, on the other hand, of order one when the difference approaches zero only if the maxima of $|\eta(x)|$ and $|\eta'(x)|$ do so simultaneously. The integrals of the calculus of variations have continuity of this latter type, and it was desired to have a theory which should apply to them as a special case. It is shown therefore in the following pages that differentials $L(\eta)$ and $B(\eta, \eta)$ with continuity of order one are expressible in the forms

$$L(\eta) = \int_a^b \eta(x)\,du(x) + \int_a^b \eta'(x)\,du_1(x),$$

$$B(\eta, \eta) = \int_a^b \int_a^b \eta(x)\eta(y)\,d_{xy}\,p(x, y) + 2\int_a^b \int_a^b \eta(x)\eta'(y)\,d_{xy}\,q(x, y) + \int_a^b \int_a^b \eta'(x)\eta'(y)\,d_{xy}\,r(x, y).$$

If the functional $F(\lambda)$ has a minimum at $\lambda_0$ then it is proved that $u$ and $u_1$ must satisfy an equation of the form

$$u_1(x) - \int_a^x u(x)\,dx = kx + l,$$

where $k$ and $l$ are constants. Furthermore, under restrictions which are explained in §§ 4 and 5, a necessary condition for a minimum analogous to the Jacobi condition of the calculus of variations is deduced. It is proved that when $F(\lambda_0)$ is a minimum the equation

$$\int_a^b [u(y)\,d_y\,q(y, x) + u'(y)\,d_y\,r(x, y)] - \int_a^x \int_a^b [u(y)\,d_y\,p(x, y) + u'(y)\,d_y\,q(x, y)]\,dx = kx + l$$

can have no solution $u(x)$, except $u(x) \equiv 0$, vanishing at $x = a$ and a point $x = x'$ between $a$ and $b$.

These results are deduced in §§ 2 and 5. In §§ 1, 3, and 4 the necessary definitions of differentials are given and their properties discussed. The interpretation of the results of the paper for the integrals of the calculus of variations is given in § 6.

## 1. Linear functionals and first differentials

If $\mathfrak{L} \equiv [\lambda]$ be a class of arcs in the plane representable in the form

$$y = \lambda(x), \qquad a \leqq x \leqq b,$$

then we mean by the functional operation $F(\lambda)$ a real, single-valued function of the curve $\lambda$ such that to every $\lambda$ in the class $\mathfrak{K}$ there corresponds a real number $F(\lambda)$.

If $\lambda_0$ is of class $C^{(n)}$,* then by the neighborhood $(\lambda_0)_\delta^n$ of order $n$ is meant the totality of arcs $\lambda$ of class $C^{(n)}$ satisfying the inequalities:
$|\lambda(x) - \lambda_0(x)| < \delta, \ |\lambda'(x) - \lambda_0'(x)| < \delta, \cdots, \ |\lambda^{(n)}(x) - \lambda_0^{(n)}(x)| < \delta.$
Consider now a class $\mathfrak{K}$ containing all arcs in a neighborhood $(\lambda_0)_\delta^n$ of an arc $\lambda_0$ of class $C^{(n)}$. A functional $F(\lambda)$ is said to have *continuity of order $n$ at $\lambda_0$* if for every $\epsilon$ there exists a $\delta$ such that the inequality

$$|F(\lambda) - F(\lambda_0)| < \epsilon$$

holds whenever $\lambda$ is in $(\lambda_0)_\delta^n$.

A functional $L(\lambda)$ is said to be linear in a class $\mathfrak{K}$ if for some constant $A$ it has the following properties:†

$$L(c_1\lambda_1 + c_2\lambda_2) = c_1 L(\lambda_1) + c_2 L(\lambda_2), \tag{1}$$

$$|L(\lambda)| \leqq A \cdot M(\lambda), \tag{2}$$

whenever $\lambda_1, \lambda_2$ and $c_1\lambda_1 + c_2\lambda_2$, with $c_1, c_2$ constants, are in $\mathfrak{K}$, and where $M(\lambda)$ denotes the maximum value of $|\lambda|$. In the class $\mathfrak{K}_0$ of arcs $\lambda$ continuous on $a \leqq x \leqq b$ a functional $L(\lambda)$ which has the property (1) and is continuous with order zero will also have the property (2).‡ F. Riesz§ has shown that such a functional is always in the form

$$L(\lambda) = \int_a^b \lambda(x)\, du(x),$$

where $u(x)$ is of limited variation on the interval $ab$, and the integral is taken in the sense of Stieltjes.

THEOREM 1. *Let $\mathfrak{K}_1$ be the totality of arcs $\lambda$ which are of class $C'$ on the interval $a \leqq x \leqq b$. If $L(\lambda)$ has the linear property* (1) *and is continuous with order* 1 *in $\mathfrak{K}$, it is always expressible, indeed in an infinity of ways, in the form*

$$L(\lambda) = \int_a^b \lambda(x)\, du(x) + \int_a^b \lambda'(x)\, du_1(x), \tag{1}$$

*where $u(x)$, $u_1(x)$ are functions of limited variation on $a \leqq x \leqq b$.*||

* An arc $\lambda(x)$ is of class $C^{(n)}$ on $a \leqq x \leqq b$, if $\lambda(x)$, $\lambda'(x)$. $\cdots$. $\lambda^{(n)}(x)$ exist and are continuous on this interval; it is of class $D^{(n)}$ if $\lambda(x)$ is continuous and consists of a finite number of arcs of class $C^{(n)}$, Cf. Bolza, *Lectures on the Calculus of Variations*, p. 7. The arc will be said to be of class $D$ if it is bounded and has a finite number of discontinuities of the first kind.

† Riesz II, p. 10.

‡ F. Riesz I, p. 974.

§ Ibid., pp. 974–977. See also Riesz II, p. 10.

|| C. A. Fischer, *Note on the order of continuity of functions of lines*, Bulletin of the American Mathematical Society, vol. 23 (1916–7), pp. 88–90.

For, from the hypothesis that $\lambda$ is of class $C'$, we may write

$$\lambda(x) = \int_a^x \lambda'(x)\,dx + \lambda(a).$$

By the property (1) of a linear functional,

$$L(\lambda) = L\left(\int_a^x \lambda'(x)\,dx\right) + L(\lambda(a)).$$

The first term on the right of the equality is linear and has continuity of order zero in the class of all functions $\lambda'(x)$ which are continuous. Hence with the help of the theorem of Riesz,

$$L(\lambda) = \int_a^b \lambda'(x)\,du_1(x) + \int_a^b \lambda(x)\,du(x),$$

where $u(x)$ is defined by the conditions,

$$u(a) = 0, \qquad u(x) = L(1) \qquad \text{for} \qquad a < x \leqq b.$$

The infinity of ways is evident from the fact that $u(x)$ and $u_1(x)$ may be altered by a constant; but there are even more representations as will be indicated at the end of § 2.

Fréchet* has given a definition of a differential of a functional $F(\lambda)$ defined on the class $\mathfrak{C}_0$ of functions $\lambda(x)$ continuous on $a \leqq x \leqq b$. According to this definition a functional $F(\lambda)$ has a differential at $\lambda_0$ if there exists a linear functional $L(\Delta\lambda)$ such that for every arc $\lambda_0 + \Delta\lambda$ in $\mathfrak{C}_0$

$$F(\lambda_0 + \Delta\lambda) - F(\lambda_0) = L(\Delta\lambda) + \epsilon(\Delta\lambda) \cdot M(\Delta\lambda),$$

where $M(\Delta\lambda)$ is the maximum of $|\Delta\lambda|$ on $a \leqq x \leqq b$, and $\epsilon(\Delta\lambda)$ is a functional which approaches zero with $M(\Delta\lambda)$. It is an immediate consequence of Fréchet's definition that $F(\lambda)$ has continuity of order zero at $\lambda_0$, a property which is not possessed by the functionals occurring in the calculus of variations. The following definition is however applicable at least to the functionals defined by the integrals of the calculus of variations containing only first derivatives.

Definition 1. *Let $\lambda_0$ be an arc of class $C'$ on $a \leqq x \leqq b$ and $F(\lambda)$ a functional defined at least in a neighborhood $(\lambda_0)'_\delta$. Then $F(\lambda)$ is said to have a differential at $\lambda_0$ if there exists a linear functional $L(\Delta\lambda)$ with continuity of order one, such that for all arcs $\lambda_0 + \Delta\lambda$ in $(\lambda_0)'_\delta$,*

$$F(\lambda_0 + \Delta\lambda) - F(\lambda_0) = L(\Delta\lambda) + \epsilon(\Delta\lambda) M_1(\Delta\lambda). \tag{2}$$

* Fréchet I, p. 139.

*$M_1(\Delta\lambda)$ is the maximum of the values of $|\Delta\lambda|$ and $|\Delta\lambda'|$ on $a \leqq x \leqq b$, and $\epsilon(\Delta\lambda)$ is a functional which vanishes with $M_1(\Delta\lambda)$.*

The linear functional $L(\Delta\lambda)$ is always expressible in the form (1), according to Theorem 1.

## 2. The first variation of $F(\lambda)$

It is proposed in this section to study the properties of the first variation, in other words the first differential, of a functional $F(\lambda)$ which has a minimum or maximum at a particular $\lambda_0$. For this purpose we shall be concerned with (1) an arc $\lambda_0$ of class $C'$ on the interval $x_1 \leqq x \leqq x_2$, joining the two fixed points $(x_1, y_1)$ and $(x_2, y_2)$; (2) a functional $F(\lambda)$ defined in a neighborhood $(\lambda_0)'_\delta$ of order one of $\lambda_0$, which has a differential of the kind described in § 1 at the arc $\lambda_0$; (3) the totality $\mathfrak{K}$ of all arcs of class $C'$ joining $(x_1\, y_1)$ with $(x_2\, y_2)$ and lying in $(\lambda_0)'_\delta$.

Definition 2. *The functional $F(\lambda)$ is said to have a minimum at $\lambda_0$ in the class $\mathfrak{K}$ if there exists a neighborhood of order one of $\lambda_0$ in which $F(\lambda) \geqq F(\lambda_0)$ for every arc $\lambda$ of $\mathfrak{K}$.*

Consider the special one-parameter family of arcs

$$y = \lambda(x, \alpha) = \lambda_0(x) + \alpha\,\eta(x),$$

for which $\eta(x)$ is of class $C'$ on $x_1 \leqq x \leqq x_2$ and $\eta(x_1) = \eta(x_2) = 0$. These will all be in $\mathfrak{K}$ for sufficiently small values of $\alpha$, and the value of $F(\lambda)$ on any one of them is from formula (2)

$$F(\lambda(x, \alpha)) = F(\lambda_0) + \alpha L(\eta) + \alpha M(\eta)\,\epsilon(\alpha\eta).$$

Then will follow readily

Lemma 1. *If $F(\lambda_0)$ is a minimum according to the definition given above, then*

$$L(\eta) = \int_{x_1}^{x_2} \eta\, du + \int_{x_1}^{x_2} \eta'\, du_1 \tag{3}$$

*must vanish for every function $\eta(x)$ of class $C'$ on $x_1\, x_2$ such that*

$$\eta(x_1) = \eta(x_2) = 0.$$

We shall next determine a necessary and sufficient condition that the sum of two integrals of form (3) shall vanish as described. Since the value of a Stieltjes integral is unaltered if the function of limited variation is changed at a finite or denumerable infinity of points between $x_1$ and $x_2$, and since the discontinuities of a function of limited variation are denumerable, we may take $u(x)$, $u_1(x)$ to be regular* for $x_1 < x < x_2$; that is, at every point between $x_1$ and $x_2$, $2\,u(x) = u(x+0) + u(x-0)$.

* Fréchet II, p. 217.

According to a well-known property of a Stieltjes integral,*

$$(4)\quad \int_{x_1}^{x_2} \eta(x)\,du(x) = \eta(x_2)u(x_2) - \eta(x_1)u(x_1) - \int_{x_1}^{x_2} u(x)\,d\eta(x),$$

and since $\eta(x)$ is of class $C'$ it follows readily from the definition of a Stieltjes integral that the last integral is also expressible as an ordinary Riemann integral

$$\int_{x_1}^{x_2} u(x)\,\eta'(x)\,dx.$$

Furthermore one may prove without difficulty the relation

$$\int_{x_1}^{x_2} \eta'(x)\,u(x)\,dx = \int_{x_1}^{x_2} \eta'(x)\,d\int_{x_1}^{x} u(x)\,dx,$$

so that with the help of the values $\eta(x_1) = \eta(x_2) = 0$ the expression for $L(\eta)$ from (3) may be written in the form

$$(5)\qquad \int_{x_1}^{x_2} [\eta(x)\,du(x) + \eta'(x)\,du_1(x)] = \int_{x_1}^{x_2} \eta'(x)\,d\hat{u}(x),$$

where

$$\hat{u}(x) = u_1(x) - \int_{x_1}^{x} u(x)\,dx.$$

Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be any four points of the interval $x_1 x_2$ such that

$$x_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < x_2, \qquad \beta_1 - \alpha_1 = \beta_2 - \alpha_2.$$

Let $h$ be a positive number such that

$$x_1 < \alpha_1 - h < \beta_1 + h < \alpha_2 - h < \beta_2 + h < x_2.$$

Define a continuous function $\eta'(x)$ by the conditions

$$\eta'(x) \equiv 0 \ \text{ for } \ x_1 \leqq x \leqq \alpha_1 - h,\ \beta_1 + h \leqq x \leqq \alpha_2 - h,\ \beta_2 + h \leqq x \leqq x_2,$$

$$\eta'(x) \equiv 1 \ \text{ for } \ \alpha_1 \leqq x \leqq \beta_1,$$

$$\eta'(x) \equiv -1 \ \text{ for } \ \alpha_2 \leqq x \leqq \beta_2,$$

and the condition that $\eta'(x)$ is linear in the remaining parts of the interval $x_1 x_2$. Then the function

$$\eta(x) = \int_{x_1}^{x} \eta'(x)\,dx$$

is of class $C'$ and has $\eta(x_1) = \eta(x_2) = 0$.

* T.-J. Stieltjes, *Récherches sur les fractions continues*, Annales de la Faculté des Sciences de Toulouse, vol. VIII (1894), J. 72.

The substitution of $\eta'(x)$ so defined in the expression (4) gives

$$\frac{1}{h}\int_{\alpha_1-h}^{\alpha_1}(x-\alpha_1+h)\,d\hat{u}+\int_{\alpha_1}^{\beta_1}d\hat{u}-\frac{1}{h}\int_{\beta_1}^{\beta_1+h}(x-\beta_1-h)\,d\hat{u}$$
$$-\frac{1}{h}\int_{\alpha_2-h}^{\alpha_2}(x-\alpha_2+h)\,d\hat{u}-\int_{\alpha_2}^{\beta_2}d\hat{u}+\frac{1}{h}\int_{\beta_2}^{\beta_2+h}(x-\beta_2-h)\,d\hat{u},$$

and this must vanish for every choice of $\alpha_1, \beta_1, \alpha_2, \beta_2, h$ satisfying the conditions described above if $F(\lambda)$ is to have a minimum at $\lambda_0$. The transformation (4) applied to this expression gives, after some simplification, the necessary condition for a minimum

$$\frac{1}{h}\int_{\beta_1}^{\beta_1+h}\hat{u}(x)\,dx-\frac{1}{h}\int_{\alpha_1-h}^{\alpha_1}\hat{u}(x)\,dx=\frac{1}{h}\int_{\beta_2}^{\beta_2+h}\hat{u}(x)\,dx-\frac{1}{h}\int_{\alpha_2-h}^{\alpha_2}\hat{u}(x)\,dx.$$

If we apply the mean value theorem to each of the above integrals and take the limit as $h$ tends to zero we obtain

$$\hat{u}(\beta_1+0)-\hat{u}(\alpha_1-0)=\hat{u}(\beta_2+0)-\hat{u}(\alpha_2-0), \tag{6}$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$ satisfy the restrictions described above.

By the definition of $\hat{u}(x)$ its discontinuities are identical with those of $u_1(x)$ and are therefore denumerable. Select $\alpha_1$ arbitrarily between $x_1$, and $x_2$ and let $\alpha_2 > \alpha_1$ be a value at which $\hat{u}(x)$ is continuous. If in the equation (6) we let $\beta_1$ and $\beta_2$ approach $\alpha_1$ and $\alpha_2$ respectively, it follows that

$$\hat{u}(\alpha_1+0)-\hat{u}(\alpha_1-0)=\hat{u}(\alpha_2+0)-\hat{u}(\alpha_2-0).$$

The second member of this equality is zero by the hypothesis that $\hat{u}(x)$ is continuous at $\alpha_2$. Hence

$$\hat{u}(\alpha_1+0)-\hat{u}(\alpha_1-0)=0,$$

and the same relation holds for $u_1(x)$ whose discontinuities are coincident with those of $\hat{u}(x)$. Since $\alpha_1$ was an arbitrarily selected point between $x_1$ and $x_2$, and since $u_1(x)$ was taken to be regular, it follows from the above that $u_1(x)$ *and also* $\hat{u}(x)$ *are continuous everywhere between* $x_1$ *and* $x_2$, *so that equation* (6) *may be written*

$$\hat{u}(\beta_1)-\hat{u}(\alpha_1)=\hat{u}(\beta_2)-\hat{u}(\alpha_2).$$

The function $\hat{u}(x)$, being continuous and of limited variation for $x_1 < x < x_2$ has a derivative everywhere except on a set of points of measure zero.* Let $\alpha_1$ be an arbitrarily selected point on $x_1 x_2$ and let $\alpha_2 > \alpha_1$ be a point where $\hat{u}(x)$ has a well defined derivative. The differences $\beta_1 - \alpha_1$ and $\beta_2 - \alpha_2$ being equal we may write

* Vallée Poussin, *Cours d'analyse infinitésimale*, 3d edition, vol. I, p. 275.

$$\frac{\hat{u}(\beta_1) - \hat{u}(\alpha_1)}{\beta_1 - \alpha_1} = \frac{\hat{u}(\beta_2) - \hat{u}(\alpha_2)}{\beta_2 - \alpha_2}. \tag{7}$$

If we let $\beta_1 \to \alpha_1$, $\beta_2 \to \alpha_2$ in such a manner that $\beta_2 - \alpha_2$ and $\beta_1 - \alpha_1$ remain equal, the right member of (7) tends to a limit and the left member therefore approaches the same limit. Hence we have the result that $\hat{u}(x)$ has a derivative everywhere between $x_1$ and $x_2$ and that this derivative has a constant value,

$$\hat{u}'(x) = k, \qquad x_1 < x < x_2.$$

We are now able to prove that $\hat{u}(x)$, and therefore $u_1(x)$, is continuous also at the points $x_1$ and $x_2$. To do this, consider the equation

$$\int_{x_1}^{x_2} \eta'(x)\, d\hat{u}(x) = 0.$$

Since $\hat{u}(x)$ is discontinuous at most at $x_1$ and $x_2$ and is linear between these values, this equation may be written

$$\eta'(x_1)[\hat{u}(x_1+0) - \hat{u}(x_1)] + \eta'(x_2)[\hat{u}(x_2) - \hat{u}(x_2-0)] + k\int_{x_1}^{x_2} \eta'(x)\, dx = 0,$$

or, since $\eta(x_1) = \eta(x_2) = 0$,

$$\eta'(x_1)[\hat{u}(x_1+0) - \hat{u}(x_1)] + \eta'(x_2)[\hat{u}(x_2) - \hat{u}(x_2-0)] = 0,$$

a result which must be true for every $\eta(x)$ of class $C'$ vanishing at $x_1$ and $x_2$. Hence it follows that

$$\hat{u}(x_1+0) = \hat{u}(x_1), \qquad \hat{u}(x_2-0) = \hat{u}(x_2),$$

and that the same relation holds for $u_1(x)$. We have accordingly proved

LEMMA 2. *If $u(x)$, $u_1(x)$ are of limited variation on the interval $x_1 x_2$ and regular for $x_1 < x < x_2$, and if the integral*

$$\int_{x_1}^{x_2} [\eta(x)\, du(x) + \eta'(x)\, du_1(x)] \tag{8}$$

*vanishes for every $\eta(x)$ of class $C'$ such that $\eta(x_1) = \eta(x_2) = 0$, then a relation of the form*

$$\hat{u}(x) \equiv u_1(x) - \int_{x_1}^{x} u(x)\, dx = kx + l \tag{9}$$

*must hold everywhere on the interval $x_1 x_2$, $k$ and $l$ being constants.*

*Conversely, if the last equation is true the integral* (8) *vanishes for all functions $\eta(x)$ with the properties just described.*

The converse follows readily by substituting $\hat{u}(x) = kx + l$ in the formula (5).

It is interesting also to find the additional conditions on $u(x)$ and $u_1(x)$ which must hold if the expression (3) for the functional $L(\eta)$ is to vanish for all functions $\eta(x)$ of class $C'$ on the interval $x_1 x_2$, whether or not the conditions $\eta(x_1) = \eta(x_2) = 0$ are satisfied. The necessary condition of the lemma just proved must be satisfied in this case also, and the relation analogous to (5) now is

$$\int_{x_1}^{x_2} [\eta(x)du(x) + \eta'(x)du_1(x)] = [\eta(x)u(x)]_{x_1}^{x_2} + \int_{x_1}^{x_2} \eta'(x)d\hat{u}(x)$$

$$= [\eta(x)u(x)]_{x_1}^{x_2} + k\int_{x_1}^{x_2} \eta'(x)dx = \eta(x_2)[u(x_2)+k] - \eta(x_1)[u(x_1)+k].$$

This last expression vanishes for every $\eta(x)$ of class $C'$; hence we conclude that

$$k = -u(x_2) = -u(x_1),$$

and it follows from the relation (9) that

$$\begin{aligned} u_1'(x_1) &= u(x_1+0) - u(x_1), \\ u_1'(x_2) &= u(x_2-0) - u(x_2). \end{aligned} \tag{10}$$

By the preceding arguments we have therefore arrived at the following theorem:

THEOREM 2. *If $F(\lambda_0)$ be a maximum or minimum according to the conditions described in Definition 2, then the functions $u(x)$ and $u_1(x)$ occurring in the first variation*

$$\int_{x_1}^{x_2} [\eta(x)du(x) + \eta'(x)du_1(x)]$$

*must satisfy the relation*

$$u_1(x) - \int_{x_1}^{x} u(x)dx = kx + l.$$

*If $F(\lambda_0)$ be a maximum or minimum with respect to the values of $F(\lambda)$ for all arcs whatsoever of class $C'$ on $x_1 \leqq x \leqq x_2$ lying in a neighborhood $(\lambda_0)'$, then the additional conditions*

$$u_1'(x_1) = u(x_1+0) - u(x_1),$$

$$u_1'(x_2) = u(x_2-0) - u(x_2)$$

*must be satisfied.*

In relations (9) and (10) we have the conditions that integral (8) shall vanish for all $\eta$'s of class $C'$ on $x_1 x_2$. Consequently, if we have two representations of the same linear functional,

$$\int_{x_1}^{x_2} [\eta du + \eta' du_1] \quad \text{and} \quad \int_{x_1}^{x_2} [\eta dv + \eta' dv_1],$$

then $u - v$ and $u_1 - v_1$ must satisfy the conditions that

$$\int_{x_1}^{x_2} [\eta\, d(u - v) + \eta'\, d(u_1 - v_1)]$$

shall vanish for all $\eta$'s of class $C'$ on $x_1 x_2$.

## 3. Bilinear functionals and second differentials

For the purpose of discussing further the conditions which characterize a minimum of the functional $F(\lambda)$ we next introduce the notion of a second differential. The basis of the investigation is found in the paper of Fréchet on bilinear functionals.*

If $\lambda(x)$, $\mu(y)$ are two continuous functions defined on the two intervals $a \leqq x \leqq b$, $c \leqq y \leqq d$, then according to Fréchet, $B(\lambda, \mu)$ is a bilinear functional of $\lambda, \mu$ if it is a linear functional of $\lambda$ for a fixed $\mu$ and a linear functional of $\mu$ for a fixed $\lambda$. The bilinear functional $B(\lambda, \mu)$ so defined has continuity of order zero simultaneously in $\lambda, \mu$ and it has also the property that there is a constant $P$ such that

$$|B(\lambda, \mu)| \leqq PM(\lambda) M(\mu).$$

By use of the theorem of Riesz on linear functionals Fréchet derives a representation of a bilinear functional by means of two iterated Stieltjes integrals. He then shows that each of these iterated integrals is equal to a double integral of the form

$$B(\lambda, \mu) = \int_a^b \int_c^d \lambda(x) \mu(y)\, d_{xy}\, p(x, y)$$
$$= \lim_{\delta \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda(x_i') \mu(y_j') \Delta_{ij}\, p(x, y),$$

where the values $x_i$ $(i = 1, \cdots, m)$ are the division points of a partition of the interval $a \leqq x \leqq b$ of norm $\delta$, the $y_j$ $(j = 1, \cdots, n)$ are division points for a similar partition of $c \leqq y \leqq d$, and where $\Delta_{ij}\, p(x, y)$ is the second difference

$$\Delta_{ij}\, p(x, y) = p(x_i y_j) - p(x_{i-1}, y_j) - p(x_i, y_{j-1}) + p(x_{i-1}, y_{j-1}).$$

The function $p(x, y)$ is of limited variation with respect to the set $(x, y)$ and with respect to $x$ and $y$ separately, the total variation with respect to the set $(x, y)$ being defined as the upper bound of the expression

$$\Big|\sum_i \sum_j \epsilon_i \epsilon_j' \Delta_{ij}\, p(x, y)\Big|,$$

where $\epsilon_i$, $\epsilon_j$ are of arbitrary signs but in absolute value equal to unity.

* Fréchet II, pp. 215–234.

THEOREM 3. *If $\lambda$, $\mu$ are of class $C'$ on the intervals $ab$, $cd$ and if $B(\lambda, \mu)$ is linear and continuous with order one in each variable when the other is fixed, then $B(\lambda, \mu)$ is expressible in the form*

$$B(\lambda, \mu) = \int_a^b \int_c^d \lambda(x)\mu(y)\, d_{xy}\, p(x, y) + \int_a^b \int_c^d \lambda'(x)\mu(y)\, d_{xy}\, q'(x, y)$$
$$+ \int_a^b \int_c^d \lambda(x)\mu'(y)\, d_{xy}\, q''(x, y) + \int_a^b \int_c^d \lambda'(x)\mu'(y)\, d_{xy}\, r(x, y),$$

*where the functions $p$, $q'$, $q''$, $r$ are of limited variation with respect to $x$, $y$ together and with respect to each separately.*

By hypothesis $\lambda$ and $\mu$ are expressible in the form

$$\lambda(x) = \int_a^x \lambda'(x)\, dx + \lambda(a), \qquad \mu(y) = \int_c^y \mu'(y)\, dy + \mu(c).$$

Then by the properties of $B(\lambda, \mu)$,

$$B(\lambda, \mu) = B\left(\int_a^x \lambda'(x)\, dx, \int_c^y \mu'(y)\, dy\right) + B\left(\int_a^x \lambda'(x)\, dx, \mu(c)\right)$$
$$+ B\left(\lambda(a), \int_c^y \mu'(y)\, dy\right) + B(\lambda(a), \mu(c)).$$

Since the first functional in the right member of the equality is linear and has continuity of order zero in $\lambda'$, $\mu'$, the second in $\lambda'$, $\mu$, the third in $\lambda$, $\mu'$, the fourth in $\lambda$, $\mu$, we have by help of the Fréchet theorem the desired form for $B(\lambda, \mu)$.

A definition of a second differential will now be given which is somewhat different from that of Fréchet.* Consider an arc $\lambda_0(x)$ which is of class $C'$ on the interval $x_1 \leqq x \leqq x_2$, and let $F(\lambda)$ be a functional which is well defined for all arcs $\lambda$ of class $C'$ on $x_1 x_2$ and lying in a neighborhood $(\lambda_0)'_\delta$ of order one of $\lambda_0$.

DEFINITION 3. *The functional $F(\lambda)$ will be said to have a second differential at $\lambda_0$ if there exist a linear functional $L(\lambda)$ and a bilinear functional $B(\lambda, \mu)$ having continuity of order one for all arcs $\lambda$, $\mu$ of class $C'$ on $x_1 x_2$ such that*

$$F(\lambda_0 + \eta) = F(\lambda_0) + L(\eta) + B(\eta, \eta) + M^2(\eta)\epsilon(\eta)$$

*for every arc $\lambda_0 + \eta$ of class $C'$ in a neighborhood of order one of $\lambda_0$. The symbol $M(\eta)$ represents the maximum of $|\eta|$, $|\eta'|$ on the interval $x_1 x_2$ and $\epsilon(\eta)$ is a functional which vanishes with $M(\eta)$.*

If we consider in particular a family of arcs of the form

$$y = \lambda_0(x) + \alpha\eta(x)$$

* Fréchet II, p. 232.

where $\eta(x)$ is of class $C'$ on the interval $x_1 x_2$ and $\eta(x_1) = \eta(x_2) = 0$. then

$$F(\lambda_0 + \alpha\eta) - F(\lambda_0) = \alpha L(\eta) + \alpha^2 B(\eta, \eta) + \alpha^2 M^2(\eta)\epsilon(\eta),$$

and the following lemma can readily be proved.

LEMMA 3. *If $F(\lambda_0)$ is a minimum in the sense described in Definition 2, then the conditions*

$$L(\eta) = 0, \qquad B(\eta, \eta) \geqq 0$$

*must hold for every function $\eta(x)$ of class $C'$ on the interval $x_1 x_2$ having*

$$\eta(x_1) = \eta(x_2) = 0.$$

## 4. THEOREMS CONCERNING INTEGRALS OF STIELTJES AND FRÉCHET

The theorems of the preceding sections on linear and bilinear functionals presuppose that the arcs considered are continuous. In the study of the second variation which follows it has been found convenient to introduce arcs which have a finite number of discontinuities of the first kind on the interval $x_1 x_2$ and to consider linear and bilinear functionals defined for such arcs and expressible as integrals of Stieltjes or of Fréchet. For this purpose it will be necessary to show the existence of the integrals under the new hypothesis and to prove the validity of certain relations, some of which have already been proved for the original definitions of the integrals.

THEOREM 4. *If $\alpha(x)$ is continuous and of limited variation on the interval $ab$, and $f(x)$ is of class $D$, then the integral*

$$\int_a^b f(x)\,d\alpha(x)$$

*exists.*

This is a special case of a theorem of G. A. Bliss.*

THEOREM 5. *If $\alpha(x)$ is continuous and of limited variation on $ab$ and $f(x)$ is of class $D$, then the integral*

$$\int_a^b \alpha(x)\,df(x)$$

*exists, and the relation*

* *A necessary and sufficient condition for the existence of a Stieltjes integral*, Proceedings of the National Academy of Sciences, vol. 3, pp. 633–637, November, 1917.

W. H. Young states and proves this theorem for a monotonic function $\alpha(x)$ in his paper *On integration with respect to a function of bounded variation*, Proceedings of the London Mathematical Society, ser. 2, vol. 13 (1914), p. 133. Young states but does not prove the theorem for a double integral for an arbitrary function of 2 variables of limited variation in the paper, *On multiple integration by parts and a second theorem of the mean*, ibid., vol. 16 (1917).

$$\int_a^b f(x)\,d\alpha(x) + \int_a^b \alpha(x)\,df(x) = f(b)\alpha(b) - f(a)\alpha(a)$$

*holds.**

THEOREM 6. *If $f(x)$ is of class $D'$ on $ab$, and*

$$A(x) = \int_a^x \alpha(x)\,dx + A(a),$$

*where $\alpha(x)$ is of limited variation on the same interval, then*

$$\int_a^b f'(x)\,dA(x) = [f(x)\alpha(x)]_a^b - \int_a^b f(x)\,d\alpha(x).$$

By definition,

$$\int_a^b f'(x)\,dA(x) = \lim \sum_k f'(\xi_k)\,[A(x_k) - A(x_{k-1})].$$

If the discontinuities of $f'(x)$ are among the division points of the interval $ab$, and the points $\xi_k$ are properly chosen, the right member of the above equation may be written in the form,

$$\lim \sum_k [f(x_k) - f(x_{k-1})]\frac{A(x_k) - A(x_{k-1})}{x_k - x_{k-1}}$$
$$= \lim \sum_k [f(x_k) - f(x_{k-1})] \cdot \alpha_k,$$

where $\alpha_k$ is a value between the maximum and minimum of $\alpha(x)$ on the interval $x_{k-1}\,x_k$. It is easily shown that the expressions

$$\sum_k \alpha_k [f(x_k) - f(x_{k-1})], \qquad \sum_k \alpha(\xi_k)[f(x_k) - f(x_{k-1})],$$

in which $\alpha(\xi_k)$ is the value of $\alpha(x)$ at an arbitrary point in the interval $x_{k-1}\,x_k$, and $\alpha_k$ a number which lies between the maximum and minimum of $\alpha(x)$ on the interval, both approach the same limit, so that

$$\int_a^b f'(x)\,dA(x) = \int_a^b \alpha(x)\,df(x).$$

From this last equality and the relation of Theorem 5 follows the desired relation of the theorem.

Consider a function $p(x, y)$ which is
(1) continuous on the square $S : a \leqq x \leqq b,\ a \leqq y \leqq b$;
(2) expressible in the form

* For a proof of this theorem see G. A. Bliss, *Integrals of Lebesgue*, Bulletin of the American Mathematical Society, 2d series, vol. 24, No. 1, pp. 1–47; H. E. Bray, *Elementary properties of the Stieltjes integral*, Annals of Mathematics, vol. 20 (1919), p. 185.

$$p(x,y) = \int_a^x p_1(x,y)\,dx + p(a,y) = \int_a^y p_2(x,y)\,dy + p(x,a),$$

where $p_1(x,y)$ and $p_2(x,y)$ are both of limited variation in $y$ uniformly with respect to $x$, and in $x$ uniformly with resepct to $y$.

The function $p_2(x,y)$, for example, is of limited variation in $x$ uniformly with respect to $y$ when there is a constant $V$ such that the total variation of $p_2(x,y)$ with respect to $x$ on the interval $ab$ is less than $V$ for every $y$ on $ab$.*

Consider a subdivision of $S$ into rectangles by abscissas $x_i$ $(i = 0, 1, \cdots, m)$ with $x_0 = a$, $x_m = b$, and ordinates $y_j$ $(j = 0, 1, \cdots, n)$ with $y_0 = a$, $y_n = b$, and let $\Delta_{ij}\,p(x,y)$ represent as before the second difference

$$\Delta_{ij}\,p(x,y) = p(x_i y_j) - p(x_i y_{j-1}) - p(x_{i-1}, y_j) + p(x_{i-1}, y_{j-1}).$$

LEMMA 4. *The function $p$ is of limited variation in the sense that the sums $\sum_{i=1}^m \sum_{j=1}^n |\Delta_{ij}\,p|$ have a common upper bound $P$ for all subdivisions of $S$ of the type described above.*

If the lemma is true the function $p$ is evidently of limited variation in the weaker sense of Fréchet also.

To prove the lemma write

$$\sum_i \sum_j |\Delta_{ij}\,p| \leqq \sum_j \int_{y_{j-1}}^{y_j} \sum_i |p_2(x_i, y) - p_2(x_{i-1}, y)|\,dy \leqq V(b-a).$$

THEOREM 7. *If $\lambda(x)$, $\mu(x)$ are of class $D$ on $a \leqq x \leqq b$ the integral*

$$\int\!\!\int_S \lambda(x)\mu(y)\,d_{xy}\,p(x,y) \tag{11}$$

*is well defined in the sense of Frechet.*

Consider a sum

$$\sigma = \sum_{ij} \lambda(\xi_i)\mu(\eta_j)\Delta_{ij}\,p,$$

where $\xi_i$ and $\eta_j$ are arbitrarily chosen values in the intervals $x_{i-1}\,x_i$ and $y_{j-1}\,y_j$ respectively. The sum of the terms of $\sigma$ belonging to a single row of rectangles has absolute value less than

$$\sum_i M^2 \int_{y_{j-1}}^{y_j} |p_2(x_i, y) - p_2(x_{i-1}, y)|\,dy \leqq M^2\,V(y_j - y_{j-1}),$$

where $M$ is the maximum of the values of $|\lambda(x)|$ and $|\mu(x)|$ on $ab$, and the sum of the terms of several rows is therefore in absolute value $\leqq M^2\,Vw$, where $w$ is the sum of the width of the rows. The same property is true for columns.

* H. E. Bray, loc. cit., p. 180.

Let $r$ and $s$ be the number of discontinuities of $\lambda(x)$ and $\mu(y)$ on $ab$ respectively. Then the discontinuities of the product $\lambda(x)\mu(y)$ occur on $r$ lines parallel to the $y$-axis and on $s$ lines parallel to the $x$-axis. If the intervals $x_{i-1}x_i$ and $y_{j-1}y_j$ all have lengths $\leqq 2\delta$, the sum of the terms of $\sigma$ corresponding to rectangles containing points of these lines of discontinuity will have absolute value not exceeding $(r+s)M^2V2\delta$. Furthermore, since $\lambda(x)$ and $\mu(y)$ are of class $D$ on $ab$, the norm $\delta$ can be chosen so small that the oscillation of the product $\lambda(x)\mu(y)$ in the rectangles containing none of its discontinuities will be less than $\epsilon$.

Consider now a sum $\sigma'$ formed by subdividing the intervals $x_{i-1}x_i$ and $y_{j-1}y_j$ which were used to form $\sigma$. Then

$$|\sigma - \sigma'| \leqq \epsilon P + 4(r+s)M^2V\delta .$$

For the first term on the right exceeds the absolute value of the difference of the parts of $\sigma$ and $\sigma'$ belonging to rectangles containing no discontinuities of $\lambda(x)\mu(y)$; and one half the second term exceeds the absolute value of the sum of the remaining terms of either $\sigma$ or $\sigma'$.

If two sums $\sigma$ and $\sigma''$ of norm $\delta$ are given, a third sum $\sigma'$ can be formed by using all of their division points, and its rectangles will be subdivisions of those of $\sigma''$ as well as those of $\sigma$. From the preceding paragraph it follows that the difference

$$|\sigma - \sigma''| \leqq |\sigma - \sigma'| + |\sigma' - \sigma''|$$

can be made less than $\epsilon'$ by taking the norm $\delta$ sufficiently small. It follows readily then by the usual arguments that the limit of $\sigma$ as $\delta$ approaches zero exists.

THEOREM 8. *If $\lambda(x)$ and $\mu(x)$ are of class $D$ on the interval $ab$ the two integrals*

$$(12) \qquad \int_a^b \lambda(x)\,d_x \int_a^b \mu(y)\,d_y\,p(x,y), \qquad \int_a^b \mu(y)\,d_y \int_a^b \lambda(x)\,d_x\,p(x,y)$$

*exist and are equal to the integral* (11) *of Theorem* 7.

Since $p(x,y)$ is continuous and of limited variation in $y$, the integral

$$\phi(x) = \int_a^b \mu(y)\,d_y\,p(x,y)$$

surely exists for every value of $x$.* It is to be proved first that it is also a continuous function of $x$. For this purpose, let the discontinuities of $\mu(y)$ on $a \leqq y \leqq b$ be enclosed in a set of intervals $\alpha_k\beta_k$ $(k = 1, \cdots, s)$ of total

* See Bliss, *A necessary and sufficient condition for the existence of a Stieltjes integral*, Proceedings of the National Academy of Sciences, vol. 3, pp. 633–637, November, 1917.

length less than $\epsilon$, and let $\beta_0 = a$, $\alpha_{s+1} = b$. Then

$$(13) \qquad \phi(x) = \sum_{k=1}^{s+1} \int_{\beta_{k-1}}^{\alpha_k} \mu(y)\, d_y\, p(x, y) + \sum_{k=1}^{s} \int_{\alpha_k}^{\beta_k} \mu(y)\, d_y\, p(x, y).$$

Each term of the first sum is a continuous function of $x$ according to a theorem of Bray,* and the second sum has an absolute value less than $MN\epsilon$, where $N$ is the maximum of $|p_2(x, y)|$ in $S$. For we have the relations

$$|p(x, y_j) - p(x, y_{j-1})| = \left| \int_{y_{j-1}}^{y_j} p_2(x, y)\, dy \right| \leqq N(y_j - y_{j-1}),$$

and hence

$$\sum_{k=1}^{s} \left| \int_{\alpha_k}^{\beta_k} \mu(y)\, d_y\, p(x, y) \right| \leqq \sum_{k=1}^{s} MN(\beta_k - \alpha_k) < MN\epsilon.$$

The continuity of $\phi(x)$ follows readily from the properties of the sums in the expression (13).

The function $\phi(x)$ is also of limited variation,† so that by the theorem of Bliss cited above the integrals (12) exist.

It remains to show that the integrals (12) are equal to (11). For a sufficiently fine $x$-partition,

$$(14) \qquad \left| \int_a^b \lambda(x)\, d\phi(x) - \sum_i \lambda(\xi_i)[\phi(x_i) - \phi(x_{i-1})] \right| < \epsilon/2.$$

But

$$(15) \qquad \sum_{i=1}^{n} \lambda(\xi_i)[\phi(x_i) - \phi(x_{i-1})] - \int_a^b \mu(y)\, d\rho(y) = 0,$$

where

$$\rho = \sum_{i=1}^{n} \lambda(\xi_i)[p(x_i, y) - p(x_{i-1}, y)]$$

is continuous and of limited variation. The integral in (15) exists and for a sufficiently fine $y$-partition

$$(16) \qquad \left| \int_a^b \mu(y)\, d\rho(y) - \sum_{j=1}^{s} \mu(\eta_j)[\rho(y_j) - \rho(y_{j-1})] \right| < \epsilon/2.$$

By adding (14), (15), (16), and writing $\rho$ out in full, we have

$$\left| \int_a^b \lambda(x)\, d\phi(x) - \sum_{ij} \lambda(\xi_i)\mu(\eta_j)\Delta_{ij}\, p(x, y) \right| < \epsilon,$$

which proves the theorem.

THEOREM 9. *If $p(x, y)$ is of limited variation in $y$ uniformly with respect to $x$ and $\eta(y)$ is continuous, then*

* Loc. cit., p. 180.

† Fréchet II, p. 229.

$$\int_a^x \int_a^b \eta(y)\, d_y\, p(x, y)\, dx = \int_a^b \eta(y)\, d_y \int_a^x p(x, y)\, dx.$$

If $\omega$ represents the maximum oscillation of $\eta(y)$ on the intervals $y_{j-1} y_j$, then

$$\left| \int_a^b \eta(y)\, d_y\, p(x, y) - \sum_{j=1}^n \eta(y_j')\,[p(x, y_j) - p(x, y_{j-1})] \right| \leqq \omega V.^*$$

Consider the relation

$$\left| \int_a^x \int_a^b \eta(y)\, d_y\, p(x, y)\, dx - \int_a^x \sum_{j=1}^n \eta(y_j')\,[p(x, y_j) - p(x, y_{j-1})]\, dx \right|$$

$$= \left| \int_a^x \left\{ \int_a^b \eta(y)\, d_y\, p(x, y) - \sum_{j=1}^n \eta(y_j')\,[p(x, y_j) - p(x, y_{j-1})] \right\} dx \right|$$

$$\leqq \int_a^x \left| \int_a^b \eta(y)\, d_y\, p(x, y) \right.$$

$$\left. - \sum_{j=1}^n \eta(y_j')\,[p(x, y_j) - p(x, y_{j-1})] \right| dx \leqq \omega V(b - a) \leqq \epsilon/2,$$

or,

$$\left| \int_a^x \int_a^b \eta(y)\, d_y\, p(x, y) - \sum_{j=1}^n \eta(y_j') \int_a^x [p(x, y_j) - p(x, y_{j-1})]\, dx \right| \leqq \epsilon/2.$$

It is true further that

$$\left| \int_a^b \eta(y)\, d_y \int_a^x p(x, y)\, dx \right.$$

$$\left. - \sum_{j=1}^n \eta(y_j') \int_a^x [p(x, y_j) - p(x, y_{j-1})]\, dx \right| \leqq \epsilon/2.$$

Hence,

$$\left| \int_a^x \int_a^b \eta(y)\, d_y\, p(x, y)\, dx - \int_a^b \eta(y)\, d_y \int_a^x p(x, y)\, dx \right| \leqq \epsilon,$$

and since these integrals are independent of $\epsilon$, we have the desired equality.

THEOREM 10. *If $\eta(x)$ is of class $D'$ on $ab$ and $p$, $q$, $r$ have the properties described for $p$, the integral*

$$(17)\quad I(\eta) = \int\int_S \{\eta(x)\,\eta(y)\, d_{xy} p + 2\eta(x)\,\eta'(y)\, d_{xy} q + \eta'(x)\,\eta'(y)\, d_{xy} r\}$$

*is well defined and a function $\eta_1(x)$ of class $C'$ can be chosen so that*

$$\eta_1(a) = \eta(a), \qquad \eta_1(b) = \eta(b), \qquad |I(\eta_1) - I(\eta)| < \epsilon,$$

*where $\epsilon$ is an arbitrarily assigned positive quantity.*

* H. E. Bray, loc. cit., p. 179.

Let the corners of the curve $\eta(x)$ be rounded off to form a function $\eta_1(x)$ in such a way that each interval in which $\eta$ differs from $\eta_1$ has length less than $2\delta$. Then the portions of $S$ in which the products $\eta(x)\eta(y)$ and $\eta_1(x)\eta_1(y)$ differ consist of strips of width $2\delta$ parallel to the $x$-axis and to the $y$-axis. If the numbers of strips parallel to the two axes are $r$ and $s$ respectively, then the proof used in Theorem 7 shows that the integrals

$$\int\int_S \eta(x)\eta(y)\,d_{xy}p, \qquad \int\int_S \eta_1(x)\eta_1(y)\,d_{xy}p$$

differ by less than $4(r+s)M^2V\delta$. A similar argument applies to the second and third parts of (17), and since $\delta$ is arbitrary the theorem is established.

In the sequel we shall be interested in the solution $u$ of a linear functional equation of the form

$$L(u;x) = kx + l, \tag{18}$$

where $k$, $l$ are constants. The functional $L(u;x)$ is supposed to be single-valued when $u(\xi)$, $x$ are given, and linear in the argument $u$. We wish to study some properties of this equation when it has a unique solution for each $k$, $l$.

THEOREM 11. *If $L(u;x) = kx + l$ has a unique solution $u(x)$ of class $C'$ for each $k$, $l$, then the equation $L(u;x) = 0$ has only the solution $u \equiv 0$, and there exist two linearly independent solutions $u_1$, $u_2$ of the original equation such that the determinant*

$$\begin{vmatrix} L'(u_1;x) & L'(u_2;x) \\ L(u_1;x) & L(u_2;x) \end{vmatrix}$$

*is different from zero.*

By hypothesis the equation (18) has a unique solution for each $k$, $l$ and therefore a unique solution for $k=0$, $l=0$. But from the properties of a linear functional it is clear that $u \equiv 0$ satisfies the equation $L(u;x) = 0$, and therefore this equation has only the solution $u \equiv 0$. Let $u_1$ be the unique solution of (18) for $k=1$, $l=0$; $u_2$ the solution for $k=0$, $l=1$. Then

$$\begin{vmatrix} L'(u_1;x) & L'(u_2;x) \\ L(u_1;x) & L(u_2;x) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ x & 1 \end{vmatrix} \neq 0.$$

Conversely, we can prove

THEOREM 12. *If $L(u;x) = 0$ has only the solution $u \equiv 0$ and if $u_1$, $u_2$ are two solutions of $L(u;x) = kx + l$ such that the determinant*

$$\begin{vmatrix} L'(u_1;x) & L'(u_2;x) \\ L(u_1;x) & L(u_2;x) \end{vmatrix}$$

*is different from zero, then the equation $L(u;x) = kx + l$ has one and only one solution for each $k$, $l$.*

Let $k_1, l_1$ be the constants corresponding to the solution $u_1$, and $k_2, l_2$ the constants corresponding to the solution $u_2$. Then

$$\begin{vmatrix} L'(u_1; x) & L'(u_2; x) \\ L(u_1; x) & L(u_2; x) \end{vmatrix} = \begin{vmatrix} k_1 & k_2 \\ k_1 x + l_1 & k_2 x + l_2 \end{vmatrix} = \begin{vmatrix} k_1 & k_2 \\ l_1 & l_2 \end{vmatrix} \neq 0.$$

Hence the equations

$$c_1 k_1 + c_2 k_2 = k,$$
$$c_1 l_1 + c_2 l_2 = l$$

can be solved uniquely for $c_1, c_2$, and $u = c_1 u_1 + c_2 u_2$ satisfies the equation (18). Moreover it is the only solution because of the hypothesis that $L(u; x) = 0$ has only the solution $u \equiv 0$.

## 5. The second variation of $F(\lambda)$

The purpose of this section is to study the second variation of a functional $F(\lambda)$. To this end we shall be concerned with

(1) An arc $\lambda_0$ of class $C'$ on the interval $x_1 \leqq x \leqq x_2$ joining the points $(x_1 y_1)$ and $(x_2 y_2)$.

(2) A functional $F(\lambda)$ defined on all arcs of class $C'$ in a neighborhood $(\lambda_0)'_\delta$ of order one, and having first and second differentials at $\lambda_0$ as in § 3, with $p, q', q'', r$ having properties described for $p$ in § 4. The functions $p$ and $r$ may be taken symmetric without loss of generality.

At the end of § 3 it was proved that if $F(\lambda_0)$ is a minimum then the second differential $B(\eta, \eta)$ must be positive or zero for every function $\eta(x)$ of class $C'$ on the interval $x_1 x_2$ such that $\eta(x_1) = \eta(x_2) = 0$.

We shall now consider the problem of minimizing the second variation. To this end we compute the first differential of $B(\eta, \eta)$. Written out in full,

$$B(\eta, \eta) = \int_{x_1}^{x_2}\int_{x_1}^{x_2} [\eta(x)\eta(y) d_{xy} p(x, y) \tag{19}$$
$$+ 2\eta(x)\eta'(y) d_{xy} q(x, y) + \eta'(x)\eta'(y) d_{xy} r(x, y)],$$

where

$$q(x, y) = \tfrac{1}{2}[q'(y, x) + q''(x, y)].$$

If we give to $\eta, \eta'$ the increments $\zeta, \zeta'$ and compute the first differential of (19), we obtain

$$2\int_{x_1}^{x_2} [\zeta(x) d\omega(x) + \zeta'(x) d\omega_1(x)],$$

where

$$\omega(x) = \int_{x_1}^{x_2} [\eta(y) d_y p(x, y) + \eta'(y) d_y q(x, y)],$$

(20)

$$\omega_1(x) = \int_{x_1}^{x_2} [\eta(y) d_y q(y, x) + \eta'(y) d_y r(x, y)].$$

From the results of § 2 it follows that *if a function* $\eta(x)$ *of class* $D'$ *with* $\eta(x_1) = \eta(x_2) = 0$ *minimize the expression* (19) *for the second variation it must satisfy the equation*

$$L(\eta; x) \equiv \omega_1(x) - \int_{x_1}^{x} \omega(x)\,dx = kx + l, \qquad x_1 \leqq x \leqq x_2.$$

We make the following hypotheses on $L(\eta; x)$ which is defined for every $\eta(x)$ of class $D'$ on $x_1 x_2$:

(1) It has a reciprocal $L_1(\eta; x)$ such that for every $\eta$ of class $C'$ on $x_1 x_2$

$$LL_1(\eta; x) = L_1 L(\eta; x) = \eta(x).$$

(2) When $\eta$ is of class $C'$ so is $L_1(\eta; x)$.

(3) There exists a constant $A$ such that

$$|L(\eta; x)| \leqq AM, \qquad |L'(\eta; x)| \leqq AM,$$

where $M$ is the maximum of $|\eta|$ and $|\eta'|$ on $x_1 x_2$.

THEOREM 13. *If* $L(\eta; x)$ *has the properties described above and if* $F(\lambda_0)$ *is a minimum then no solution* $u(x)$ *of class* $C'$ *of the equation*

$$L(u; x) = kx + l \tag{21}$$

*can exist vanishing at* $x_1$ *and a point* $x_1'$ *between* $x_1$ *and* $x_2$ *but not identically zero between* $x_1$ *and* $x_1'$, *and having* $u'(x_1') \neq 0$.

We shall first prove the

LEMMA 5. *If* $u(x)$ *is a solution of equation* (21) *with the properties described in the theorem, then for the function*

$$\begin{aligned} \eta(x) &= u(x), \qquad & x_1 \leqq x \leqq x_1', \\ &\equiv 0, & x_1' \leqq x \leqq x_2, \end{aligned}$$

*the value of* $B(\eta, \eta)$ *is zero.*

By Theorem 8 of § 4 the second variation can be written in the form

$$\begin{aligned} B(\eta, \eta) = &\int_{x_1}^{x_2} \eta(x)\,d_x \int_{x_1}^{x_2} [\eta(y)\,d_y p(x, y) + \eta'(y)\,d_y q(x, y)] \\ &+ \int_{x_1}^{x_2} \eta'(x)\,d_x \int_{x_1}^{x_2} [\eta(y)\,d_y q(y, x) + n'(y)\,d_y r(x, y)] \\ = &\int_{x_1}^{x_2} [\eta(x)\,d\omega(x) + \eta'(x)\,d\omega_1(x)]. \end{aligned}$$

Substitute for $\eta(x)$ the function defined in the theorem. Then

$$\begin{aligned} B(\eta, \eta) &= \int_{x_1}^{x_1'} [u(x)\,d\omega(x) + u'(x)\,d\omega_1(x)] \\ &= [u(x)\,\omega(x)]_{x_1}^{x_1'} + \int_{x_1}^{x_1'} u'(x)\,d\left[\omega_1(x) - \int_{x_1}^{x} \omega(x)\,dx\right]. \end{aligned}$$

The terms outside the integral sign vanish since $u(x_1) = u(x_1') = 0$. By hypothesis $u(x)$ satisfies the equation (21), and therefore

$$\int_{x_1}^{x_1'} u'(x)\,d\left[\omega_1(x) - \int_{x_1}^{x} \omega(x)\,dx\right] = k\int_{x_1}^{x_1'} u'(x)\,dx = k\cdot u(x)\Big]_{x_1}^{x_1'} = 0,$$

which proves the lemma.

Hence the function $\eta(x)$ thus defined gives the second variation the value zero, and this is the smallest possible value for $B(\eta, \eta)$ if $F(\lambda)$ is a minimum at $\lambda_0$, as has been proved. We can show, however, that in case there exists such a function $\eta(x)$ making $B(\eta, \eta)$ vanish, then there will surely be others which make it negative. For this purpose write

$$B(\eta, \eta) = \int\int_S \{\eta(x)\eta(y)\,d_{xy}\,p^* + 2\eta(x)\eta'(y)\,d_{xy}\,q + \eta'(x)\eta'(y)\,d_{xy}\,r\} - h\int_{x_1}^{x_2} \eta^2(x)\,dx$$

$$\equiv B^*(\eta, \eta) - h\int_{x_1}^{x^2} \eta^2(x)\,dx,$$

with $p^*$ defined by the conditions,

$$\begin{aligned} p^*(x, y) &= p(x, y) + h(y - x) \qquad &\text{for} \qquad x_1 \leqq y \leqq x, \\ &= p(x, y) &\text{for} \qquad x \leqq y \leqq x_2. \end{aligned}$$

If we denote by $L^*(\eta; x)$ the result obtained by replacing in $L(\eta; x)$ the function $p(x, y)$ by $p^*(x, y)$ we have the relation

$$L^*(\eta; x) = L(\eta; x) - h\int_{x_1}^{x}\int_{x_1}^{x} \eta(y)\,dydx.$$

LEMMA 6. *For sufficiently small values of h the equation*

$$L^*(u; x) = kx + l \tag{22}$$

*has a solution* $u_1(x, h)$ *of class* $C'$ *on* $x_1 x_2$ *corresponding to* $k = 1$, $l = 0$, *and a solution* $u_2(x, h)$ *of class* $C'$ *on* $x_1 x_2$ *corresponding to* $k = 0$, $l = 1$. *These solutions are continuous in a domain* $x_1 \leqq x \leqq x_2$, $|h| \leqq \delta$, *and are linearly independent.*

According to hypothesis (1) the equation

$$L(u; x) = v(x), \tag{23}$$

where $u$, $v$ are of class $C'$ on $x_1 x_2$ has a unique solution $u(x)$. If we apply the operation $L_1$ to the members of the equation

$$L^*(u; x) = v(x), \tag{24}$$

we obtain

$$u(x, h) - hL_1\left(\int_{x_1}^{x}\int_{x_1}^{x} u(y)\,dydx; x\right) = L_1(v; x) = \phi(x),$$

or,

$$u(x, h) = \phi(x) + hL_2(u; x), \tag{25}$$

where $L_2$ has the properties (2) and (3) of $L_1$ and $\phi(x)$ is of class $C'$. Repeated applications of the last equation give the series

$$u(x, h) = \phi(x) + hL_2(\phi; x) + h^2 L_2^2(\phi; x) + \cdots + h^n L_2^n(\phi; x) + \cdots. \tag{26}$$

By property (2) each term of this series is of class $C'$. By (3) the terms of the series formed of the $x$-derivatives of (26) as well as the terms of the series itself are dominated by the terms of the series,

$$M(1 + hA(x_2 - x_1)^2 + h^2 A^2(x_2 - x_1)^4 + \cdots + h^n A^n(x_2 - x_1)^{2n} + \cdots),$$

whence it is seen that the two series are uniformly convergent for $x_1 \leqq x \leqq x_2$, $|h| \leqq \delta$, when $\delta$ is sufficiently small.

The series (26) satisfies equation (25) and therefore (24) as we see by operating on (26) with $L_2$. To justify this last statement write (26) in the form $u = s_n + r_n$, where $s_n$ is the sum of the first $n$ terms. Then

$$|L_2(u) - L_2(s_n)| = |L_2(r_n)| \leqq \epsilon'.$$

Also (26) is the only solution. For if there were two solutions, their difference $\psi(x)$ would satisfy the equation

$$\psi(x) = hL_2(\psi; x). \tag{27}$$

This equation can have no solution other than $\psi \equiv 0$ for $h \leqq \delta$ when $\delta$ is sufficiently small. For the functions $h^n L_2^n(\phi; x)$ in (26) tend to zero if $|h| \leqq \delta$; but if $\psi(x, h)$ satisfies (27) all these terms are identical with $\phi(x) \equiv 0$, and therefore $\psi \equiv 0$.

These results together with Theorem 11 of § 4 prove the lemma.

Lemma 7. *If equation* (21) *has a solution* $u(x)$ *as described in the theorem then for sufficiently small values of* $|h|$ *the equation* (22) *has a solution* $u(x, h)$ *with similar properties.*

Let $u(x)$ be the solution of (21) with constants $k, l$. Then by Theorems 11 and 12 of § 4 it is expressible in the form

$$u(x) = ku_1(x, 0) + lu_2(x, 0).$$

Since $u(x)$ vanishes at $x_1$ it may be that both $u_1(x_1, 0)$ and $u_2(x_1, 0)$ are zero; in which case

$$u(x, h) = ku_1(x, h) + lu_2(x, h)$$

is the desired solution. If $u_1(x_1, 0)$ and $u_2(x_1, 0)$ are not both zero, then

$$u(x_1) = ku_1(x_1, 0) + lu_2(x_1, 0) = 0,$$

or,

$$k : l = -u_2(x_1, 0) : u_1(x_1, 0);$$

and

$$u(x, h) = \begin{vmatrix} u_1(x, h) & u_2(x, h) \\ u_1(x_1, h) & u_2(x_1, h) \end{vmatrix}$$

is the solution demanded. For, from the hypothesis that $u'(x_1')$ is different from zero it follows that $u_1(x_1' - \delta)$ and $u_1(x_1' + \delta)$ will have opposite signs for a sufficiently small $\delta$. The same will be true of $u(x_1' - \delta, h)$ and $u(x_1' + \delta, h)$ for sufficiently small $|h|$. Hence $u(x, h)$ for such values of $h$ will surely vanish at least once between $x_1' - \delta$ and $x_1' + \delta$. The value $x_h$ below can be selected as the first zero of $u(x, h)$ after $x_1' - \delta$.

To prove Theorem 13, choose $h > 0$, and

$$\begin{aligned} \eta(x) &= u(x, h) \qquad \text{for} \qquad x_1 \leqq x \leqq x_h, \\ &\equiv 0 \qquad\qquad\quad \text{for} \qquad x_h \leqq x \leqq x_2, \end{aligned}$$

where $x_h$ is a zero of $u(x, h)$ between $x_1$ and $x_2$. Then

$$B(\eta, \eta) = B^*(\eta, \eta) - h\int_{x_1}^{x_2} \eta^2(x)\,dx = -h\int_{x_1}^{x_2} \eta^2(x)\,dx < 0,$$

since $B^*(\eta, \eta) = 0$ for the $\eta$ just chosen. Finally, if $B(\eta, \eta) < 0$ for an $\eta$ of class $D'$ it will also be less than zero for an arc of class $C'$ as is seen by applying Theorem 10 of § 4.

## 6. Application to the calculus of variations

The purpose of this part of the paper is to interpret the foregoing results in the case of the functionals of the calculus of variations.

For the simplest problem of the calculus of variations the functional $F(\lambda)$ has the form

(28) $$F(\lambda) = \int_{x_1}^{x_2} f(x, \lambda, \lambda')\,dx.$$

We have seen that the functions $u(x)$ and $u_1(x)$ occurring in the expression (3) of the first variation of $F(\lambda)$ are not uniquely determined by the $F(\lambda)$, since the integrals are unaltered if $u(x)$ and $u_1(x)$ are each increased or diminished by a constant; hence we are at liberty to assign to them an arbitrary value, say zero, at a particular point of the interval $x_1 x_2$. We may then make $u(x)$ and $u_1(x)$ vanish at $x_1$. If for the functional $F(\lambda)$ in (28) we define $u(x)$ and $u_1(x)$ as follows:

(29) $$u(x) = \int_{x_1}^{x} f_\lambda\,dx, \qquad u_1(x) = \int_{x_1}^{x} f_{\lambda'}\,dx,$$

then the expression (3) for its first variation assumes the familiar form

$$\int_{x_1}^{x} (f_\lambda \eta + f_{\lambda'} \eta')\, dx .$$

We have derived as a first necessary condition for a minimum of $F(\lambda)$ the relation

$$\hat{u}'(x) = k ,$$

where

$$\hat{u}(x) = u_1(x) - \int_{x_1}^{x} u(x)\, dx .$$

For the functions in (29) this condition is equivalent to the relation

$$f_{\lambda'} - \int_{x_1}^{x} f_\lambda \, dx = k ,$$

which upon differentiation leads to the Euler equation

$$f_\lambda - \frac{d}{dx} f_{\lambda'} = 0 .$$

Since the function $u(x)$ defined in (29) is continuous the conditions (10) on $u(x)$ and $u_1(x)$, which were found to hold if the first variation vanishes for every $\eta(x)$ of class $C'$ whether or not $\eta(x_1) = \eta(x_2) = 0$, become the transversality conditions

$$f_{\lambda'}(x_1) = f_{\lambda'}(x_2) = 0 .$$

Let us next consider the second variation $B(\eta, \eta)$ of $F(\lambda)$ when $F(\lambda)$ is given by (28). Again, the functions $p, q, r$, occurring in $B(\eta, \eta)$ are not uniquely determined by $F(\lambda)$, for the value of the double integral (19) is unaltered by the addition of an arbitrary function of $x$ alone or an arbitrary function of $y$ alone. The functions may therefore be chosen identically zero for a particular value of $x$ and a particular value of $y$. Define the function $p(x, y)$ by the conditions

$$(30) \qquad \begin{aligned} p(x, y) &= \int_{x_1}^{y} P dy \qquad \text{for} \qquad x_1 \leqq y \leqq x , \\ &= \int_{x_1}^{x} P dy \qquad \text{for} \qquad x \leqq y \leqq x_2 , \end{aligned}$$

where $P = f_{\lambda\lambda}$, and make similar definitions for $q(x, y)$ and $r(x, y)$ in terms of $Q = f_{\lambda\lambda'}$ and $R = f_{\lambda'\lambda'}$, respectively. The functions $p, q, r$ so defined have all the properties assumed for $p$ in § 4.

If the functions defined above are substituted in the expression for $B(\eta, \eta)$ the double integral (19) is reduced to the single integral

$$\int_{x_1}^{x_2} (\eta^2 P + 2\eta\eta' Q + \eta'^2 R)\, dx.$$

The first member of the equation

$$L(\eta; x) = \omega_1(x) - \int_{x_1}^{x} \omega(x)\, dx = kx + l, \tag{21}$$

where $\omega$, $\omega_1$ are defined by the relations (20), assumes for the $p$, $q$, $r$ in (30) the form

$$\begin{aligned}\omega_1(x) - \int_{x_1}^{x} \omega(x)\, dx &= \int_{x_1}^{x} \{n(y)[Q(y) - (x-y)P(y)]\,dy \\ &\qquad + \eta'(y)[R(y) - (x-y)Q(y)]\,dy\} \\ &= \int_{x_1}^{x} \eta(y)\{(x-y)[Q'(y) - P(y)] \\ &\qquad - R'(y)\}\,dy + \eta(x)R(x).\end{aligned}$$

Accordingly, the equation (21) reduces to a Volterra integral equation of the first kind and has therefore a unique solution for every $v(x)$ of class $C'$, provided $R(x) \neq 0$. The properties assumed for $L(\eta; x)$ in § 5 are here justifiable. Equation (21) differentiated twice gives the Jacobi differential equation,

$$\eta(x)[Q'(x) - P(x)] + \frac{d}{dx}[\eta'(x)R(x)] = 0.$$

## 7. The character of the operation $L(u; x)$

The analog of the Jacobi condition of the calculus of variations which has been obtained as a second necessary condition for a minimum of $F(\lambda)$ is expressed in terms of a solution of the equation

$$\begin{aligned}&\int_{x_1}^{x_2} [u(y)\,d_y q(y, x) + u'(y)\,d_y r(x, y)] \\ &\qquad - \int_{x_1}^{x}\int_{x_1}^{x_2} [u(y)\,d_y p(x, y) + u'(y)\,d_y q(x, y)]\,dx = kx + l.\end{aligned} \tag{21'}$$

The first member of this equation is a linear functional $L(u; x)$ and with the help of the theorems of § 4 is expressible as a Stieltjes integral. Theorem 6 and the conditions $u(x_1) = u(x_2) = 0$ enable us to replace the two integrals

$$\int_{x_1}^{x_2} u'(y)\,d_y r(x, y), \qquad \int_{x_1}^{x_2} u'(y)\,d_y q(x, y)$$

by the integrals

$$-\int_{x_1}^{x_2} u(y)\,d_y r_2(x, y), \qquad -\int_{x_1}^{x_2} u(y)\,d_y q_2(x, y),$$

where $r_2(x, y)$ and $q_2(x, y)$ have the properties defined in § 4. Now apply Theorem 9 and we see that equation (21′) reduces to the form

$$\int_{x_1}^{x_2} u(y)\, d_y K(x, y) = kx + l,$$

where

$$K(x, y = q(y, x) - r_2(x, y) - \int_{x_1}^{x} [p(x, y) - q_2(x, y)]\, dx.$$

If the only discontinuity of the function $r_2(x, y)$ occurs on the diagonal of the square $S$ and is there equal to $R(x)$, the equation will have the form

$$(31) \qquad \int_{x_1}^{x_2} u(y)\, d_y K_1(x, y) - R(x) u(x) = kx + l,$$

where $K_1(x, y)$ is continuous and of limited variation in each variable uniformly with respect to the other.

We can show that the integral in (31) represents a transformation which has the property of complete continuity.* A transformation is said to be completely continuous if it transforms a bounded sequence into a "compact" sequence, that is, into a sequence such that every subsequence of itself contains a further subsequence which is uniformly convergent. A necessary and sufficient condition that a bounded sequence $\{\phi_n\}$ be compact is that for a positive $\epsilon$ there exists a $\delta$ such that for $|x - x'| < \delta$ and for all $\phi_n$ the inequality

$$|\phi_n(x) - \phi_n(x')| < \epsilon$$

holds.†

Let $\{u_n\}$ be a bounded sequence of continuous functions such that $|u_n| < G$. The integral

$$\phi(x) = \int_{x_1}^{x_2} u(y)\, d_y K_1(x, y)$$

is a continuous function of $x$.‡ The total variation with respect to $y$ of the function $\{K_1(x, y) - K_1(x', y)\}$ is the upper bound in the set of continuous functions $u(y)$ of the expression§

$$\frac{\left| \int_{x_1}^{x_2} u(y)\, d_y [K_1(x, y) - K_1(x', y)] \right|}{M(u)}.$$

* F. Riesz, *Ueber lineare Funktionalgleichungen*, Acta Mathematica, vol. 41: 1 (1916), p. 73.

† C. Arzelà, *Sulle funzioni di linee*, Memorie d. R. Accad. d. Scienze di Bologna, ser. 5, vol. V (1895), S. 225–244. F. Riesz, loc. cit., p. 93.

‡ H. E. Bray, loc. cit., p. 180.

§ F. Riesz, *Sur certains systèmes singuliers d'équations intégrales*, Annales scientifiques de l'école normale supérieure, vol. 28 (1911), p. 43; Fréchet II p. 217.

But on account of the continuity of $\phi(x)$ this last expression may be made less than $\epsilon/G$ by taking $|x - x'| < \delta$. Hence the total variation with respect to $y$ of $\{K_1(x, y) - K_1(x', y)\}$ is less than $\epsilon/G$ for $|x - x'| < \delta$. We have then the relation

$$|\phi_n(x) - \phi_n(x')| = \left| \int_{x_1}^{x_2} u_n(y)\, d_y [K_1(x, y) - K_1(x', y)] \right| < \epsilon.$$

F. Riesz discusses in his article on linear functional equations referred to above the inversion of a transformation of the form $E - A$, where $E$ is the identical transformation and $A$ is a completely continuous transformation. The results of his article are applicable to the equation (31) if $R(x) \neq 0$.

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